Closed-Form Solutions for Feedback Control with Terminal Constraints

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The problem of closed-loop control of maneuvers between two states for linear dynamical systems, subject to an arbitrarily specified terminal state, is considered. The feedback controller design is based on finite-time quadratic regulator theory. Closed-form expressions for the optimal control law are developed. Solutions are presented for both conventional and smoothed control profiles with fixed and/or free end condition problems. In the maneuvers using control-rate penalties, smooth profiles are generated throughout the maneuvers in the sense that the initial condition jump discontinuities have been eliminated. Several examples involving large-angle maneuvers of a spacecraft are demonstrated. Results include control maneuvers from one state to another, such as rest to rest and spin to rest, which effectively justify the solutions developed in this paper.

Introduction

MANY applications of dynamical systems including spacecraft, aircraft, and chemical processes involve maneuvering between two states during some phases of a typical mission. In general, the terminal states can be either at rest or moving as required by individual mission objectives. The requirement for maneuvering strategies has motivated the development of a number of different open-loop and closed-loop techniques for dynamical systems, based on optimal control theory.[1]

Two articles[2,3] in particular exist and a fair amount of success has been obtained for the application of open- and closed-loop methods to large flexible spacecraft. Breakwell[2] has proposed a formulation for small angle maneuvers of a spacecraft based on the remaining time to maneuver. Turner and Chun[3] have addressed a control-rate penalty technique for smoothing the control of large-angle maneuvers for a spinning spacecraft. Both papers weigh the terminal states in the cost function for the closed-loop technique. The drawback, as is common with penalized terminal states, is the numerical difficulty associated with a very large terminal weighting matrix, which precludes effective solution when some terminal states are required to be very accurate. Repeated simulation of a complex system over various terminal weighting matrix ranges can be very costly and even infeasible.

The optimal control problem in this paper is specified by defining a performance index, which consists of an integral of quadratic forms in the state, control, and control-rate subject to prescribed terminal constraints. The terminal constraints replace the weighted quadratic form in the terminal states. The necessary conditions, which result from minimizing the performance index, lead to three nonlinear Riccati-like matrix differential equations.[1] One equation is the standard Riccati equation, which possesses a well known solution in terms of steady-state plus transient term[3-5] when the system is completely observable. The remaining two equations are coupled to the standard Riccati equation. Closed-form solutions are developed for these two equations. The scheme is based on the success of matrix transformations which reconstruct the three coupled nonlinear matrix differential equations into uncoupled linear matrix differential equations in terms of new variable matrices. The computational burden of numerical integrations thus has been bypassed when this technique with closed-form solution is implemented, thereby permitting numerical simulations to be carried out at substantial computational savings. The control profile thus obtained would bring the system from its initial state to the arbitrarily specified final state within a certain time period. The prescribed terminal state is to be satisfied with zero error in contrast to the more common situation where the terminal errors are required to be only approximately zero.[2,3]

Large-angle, single-axis maneuvers of a space vehicle consisting of a rigid hub with four identical elastic appendages attached symmetrically about the hub (Fig. 1) are presented as examples. The structure is required to rotate about an axis perpendicular to the plane of the appendages, and only in-plane elastic displacements of the flexible members are assumed. The control system consists of a single controller in the rigid hub and four appendage torque controllers, one located half-way along the space of each appendage. The

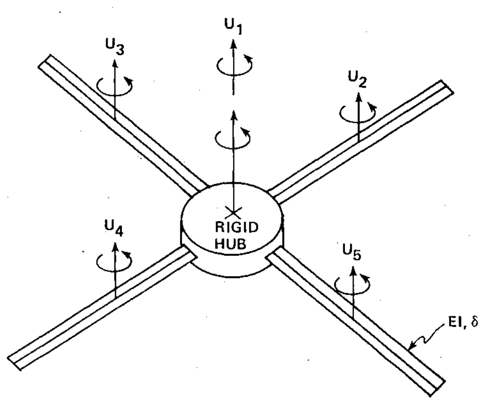

Fig. 1 Undeformed structure. $U_1, U_2, ..., U_5$ **≡ set of distributed control torques.**

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equations shown in this paper are valid for the general case of multiple controllers on each appendage. For simplicity, however, in the example maneuvers presented only the single appendage controller case is considered. In addition, since all the elastic appendages are assumed to have identical initial conditions, the four appendage controllers produce the same control torque, and the structure as a whole can be assumed to exhibit only antisymmetric deformation modes (Fig. 2).

Six different test cases are illustrated for control maneuvers between two states including rest to rest, free angle spin-up, spin to rest, etc. In particular, Case 6 presents results of the sensitivity of the controlled system to structural parameter variations for the rest to rest maneuver.

The first part of the paper summarizes the basic optimal control law for a terminal controller and presents the closed-form solutions for control gains. Illustrative examples provide insight into the nature of solutions.

Optimal Terminal Controller

The optimal terminal control problem is formulated by finding the control inputs $u(t)$ to minimize

$$J=\frac{1}{2}\int_{t_o}^{t_f}(x^TF^TQFx+u^TRu)\,\mathrm{d}t \tag{1}$$

for the system

$$\dot{x}=Ax+Bu, \quad \text{given} \quad x(t_o) \tag{2}$$

with outputs

$$y=Fx \tag{3}$$

which is subject to the specified terminal constraints

$$x_i(t_f)=\bar{x}_i, \qquad i=1,\ldots,q; \quad q\le n \tag{4}$$

where x is the state vector, u is the control vector, A is the system dynamics matrix, B is the control influence matrix, F is the measurement influence matrix, $Q=Q^T\geq 0$ is the output weighting matrix, and $R=R^T>0$ is the control weighting matrix. (For maneuvers where the state is augmented by the control and control rate penalties, the matrices A and B are modified as shown in Ref. 3.) It is assumed that the system, Eq. (2), is completely observable. Of particular interest is the fact that the performance index of Eq. (1) does not contain a terminal weight matrix, which penalizes the final values of the state. It is the intent of this paper to present a numerical technique which enforces compliance with the constraints of Eq. (4), without incurring the numerical difficulties associated with a very large terminal weighting matrix approach for the problem.[3]

As shown in Ref. 1, the necessary conditions defining the optimal solution are given by the following coupled Riccati-like matrix differential equations

$$\dot{P}+PA+A^TP-PBR^{-1}B^TP+F^TQF=0; \qquad P(t_f)=0 \tag{5}$$

$$\dot{S}+(A^T-PBR^{-1}B^T)S=0 \quad ; \qquad S^T(t_f)=(\partial\psi/\partial x)\,|\,t_f \tag{6}$$

$$\dot{G}=S^TBR^{-1}B^TS; \qquad G(t_f)=0 \tag{7}$$

where $\psi^T=[\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_q]$ and the optimal continuous feedback law is given by

$$u(t)=-C(t)x(t)-D(t)\psi \tag{8}$$

and

$$C=R^{-1}B^T(P-SG^{-1}S^T) \tag{9}$$

$$D=R^{-1}B^TSG^{-1} \tag{10}$$

The control vector u will take the state vector from $x(t_0)$ at time t_0 to $x(t_f)$ at time t_f while minimizing the cost function of Eq. (1).

In this paper we present closed form solutions for Eqs. (5)-(7), thus reducing the solution for the control problem to the direct computation of algebraic equations without numerical integration. The solution for Eq. (5) has been previously obtained in Ref. 3-5 and can be written as

$$P(t)=P_{ss}+Z^{-1}(t) \tag{11}$$

where P_{ss} is the positive definite solution for the algebraic Riccati equation

$$-A^TP_{ss}-P_{ss}A+P_{ss}BR^{-1}B^TP_{ss}-F^TQF=0 \tag{12}$$

The differential equation for $Z(t)$ is given by

$$\dot{Z}==\bar{A}Z+Z\bar{A}^T-BR^{-1}B^T \tag{13}$$

with the solution

$$Z(t)=Z_{ss}-e^{\bar{A}(t-t_f)}(Z_{ss}+P_{ss}^{-1})e^{\bar{A}^T(t-t_f)} \tag{14}$$

where $e(\cdot)$ is the exponential matrix, $\bar{A}=A-BR^{-1}B^TP_{ss}$ is a stability matrix with eigenvalues having negative real parts, and Z_{ss} satisfies the algebraic Lyapunov equation

$$\bar{A}Z_{ss}+Z_{ss}\bar{A}^T=BR^{-1}B^T \tag{15}$$

The new solution for the rectangular time-varying matrix $S(t)$ in Eq. (6) follows on assuming the product form solution

$$S(t)=Z^{-1}(t)S_c(t) \tag{16}$$

Substitution of Eq. (16) into Eq. (6) with the aid of Eq. (13) leads to the linear matrix differential equation for $S_c(t)$

$$\dot{S}_c(t)-\bar{A}S_c(t)=0; \qquad S_c(t_f)=-P_{ss}^{-1}S(t_f) \tag{17}$$

The solution for $S_c(t)$ is obviously

$$S_c(t)=-e^{\bar{A}(t-t_f)}P_{ss}^{-1}S(t_f) \tag{18}$$

Now from Eq. (7), making use of Eqs. (6) and (13), one obtains the following solution for $G(t)$

$$G(t)=S^T(t)Z(t)S(t)+S^T(t_f)P_{ss}^{-1}S(t_f) \tag{19}$$

which can be easily verified by direct differentiation of Eq. (19).

We are in a position to implement those solutions for the terminal controller with any given dimension (where by the

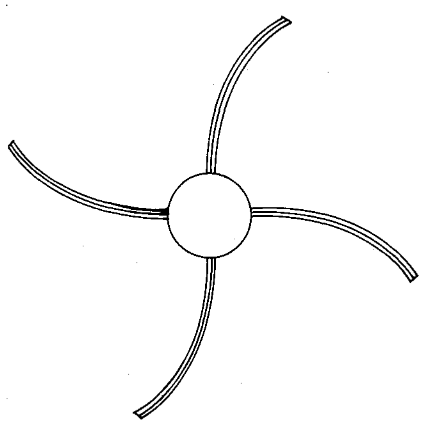

Fig. 2 Antisymmetric deformation.

dimension is meant the number of the states n and the terminal constraint q). First, we obtain the solutions (14), (16), (18) and (19), at time $t<t_f$. Then Eqs. (11), (16), and (19) will determine gains C and D in Eqs. (9) and (10). The control $u(t)$ is thus calculated from Eq. (8) at any time t. Efficient methods for the computation of $\exp[\bar{A}(t-t_f)]$ and Z^{-1} are needed to reduce the computational burden on costs and numerical errors.

The exponential matrix $\exp[\bar{A}(t-t_f)]$ is propagated forward in time using the relation

$$\exp[\bar{A}(t+\Delta t-t_f)]=\exp[\bar{A}(t-t_f)]\exp[\bar{A}\Delta t] \tag{20}$$

where $\exp[\bar{A}(t_0-t_f)]$ and $\exp[\bar{A}\Delta t]$ are computed only once, using a Padé series expansion approximation which is highly efficient and accurate.[6] When time proceeds, the matrix exponential at any other time will simply be the multiplications of the result at the previous time step. More detailed information can be found in the following section of numerical examples.

The inverse of the time-dependent symmetric matrices $Z(t)$ and $G(t)$ becomes the most time consuming operation for the terminal controller. As far as computational speed is concerned, factorization methods developed in Refs. 7 and 8 are appropriate for use.

Illustrative Examples

The specific model considered in this Paper (see Figs. 1 and 2) consists of a rigid hub with four identical elastic appendages attached symmetrically about the central hub. In particular, we consider the following idealizations: a) single-axis maneuvers; b) in-plane motion; c) antisymmetric deformations; d) small linear flexural deformations; e) linear time-invariant form of the equations of motion; and f) control actuators, modeled as concentrated torque generating devices. The distributed control system for the vehicle consists of a single controller in the rigid part of the structure, and each elastic appendage is assumed to have an arbitrary number of uniformly distributed controllers along its span. Moreover, in the optimal control performance index, the control weighting matrices are adjusted in order to have the rigid body control provide the primary torque for maneuvering the vehicle, while the appendage controllers act primarily as vibration suppressors.

For all cases, the following configuration parameters are given: the moment of inertia of the undeformed structure about the spin axis, I, is 6764 kg/m^2; the mass/length of the four identical elastic appendages, ρ, is 0.04096 kg/m; the length of each antilevered appendage, L, is 35 m; the flexural rigidity of each cantilevered appendage, EI, is 1500 Nm^2; and the radius of the rigid hub, r, is 1 m. For simplicity, we assume that each appendage has one controller located halfway along the span. In the integrations over the mass and stiffness distributions, the radius of the hub is not neglected in comparison to the appendage length. We have adopted as "assumed modes" the comparison functions

$$\phi_p(x-r)=1-\cos[p\pi(x-r)/L]$$
$$+0.5(-1)^{p+1}[p\pi(x-r)/L]^2 \qquad (p=1,2,...,n) \tag{21}$$

which satisfy the geometric and physical boundary conditions

$$\phi_p\,|_{x=r}=\phi_p'\,|_{x=r}=|\phi_p''\,|_{x=r+L}=\phi_p'''|_{x=r+L}=0 \tag{22}$$

of a clamped free appendage. In addition, full-state feedback for the control law is assumed for the results of this section.

The output vector is assumed to be given by $y=\{y_1\underline{y}_2^{kT}y_3y_4\underline{y}_5^T\}^T$, y_1 = central hub angular velocity, $\underline{y}_2$ = four times the angular velocity at each appendage controller location relative to the central hub, y_3 = hub angular position, y_4 = four times the tip deflection of each appendage, and $\underline{y}_5$ = controls and control-rates.

The number of elements in $\underline{y}_2$ is given by the number of controllers on each of the four appendages. The output sub-

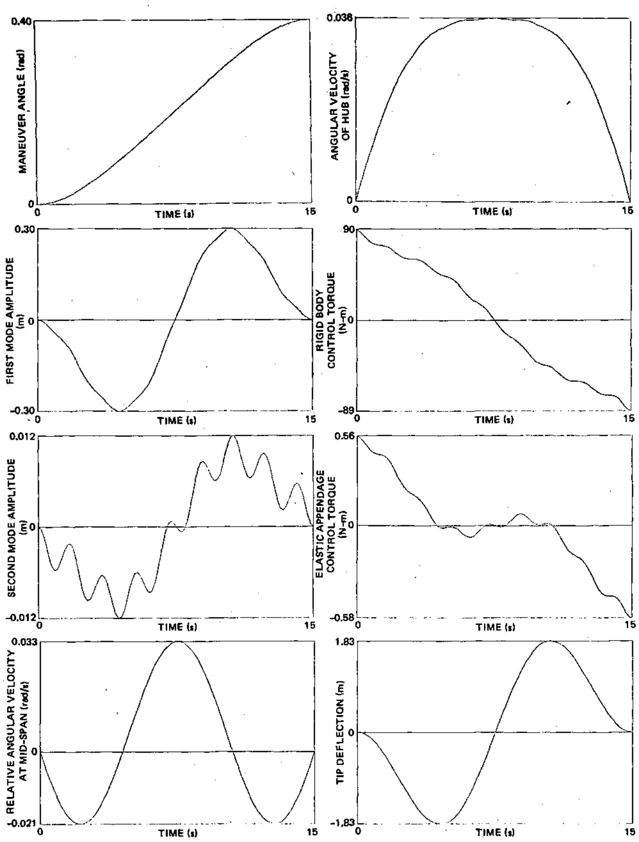

Fig. 3 Case 1, rest-to-rest maneuver, 2 modes, 5 controls.

Table 1 Test case maneuver descriptions

Case no.	No. of modes	t_f-t_0, s	k^a	θ_0, rad	$\dot{\theta}_0$, rad	θ_f, rad	$\dot{\theta}_f$, rad/s	CPU,[b] s
1	2	15	0	0	0	0.4	0	34
2	2	15	0	0	0	–	0.04	32
3	2	15	2	0	0	0.4	0	83
4	2	15	2	0	0	–	0.04	77
5	2	15	2	0.15	−0.02	0.4	0	83
6	2	15	2	0	0	0.4	0	83

[a] k is the highest order of the time-derivatives of the control torques which are penalized in the performance index. [b] The CPU time is obtained using an AMDAHL 470-V8 with 150 time steps for each integration.

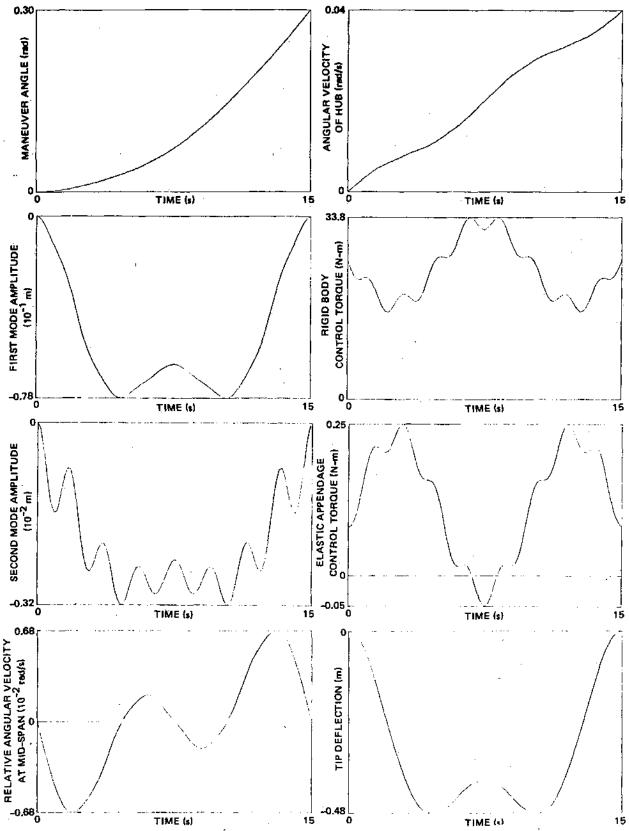

Fig. 4 Case 2, free final angle spin-up maneuver, 2 modes, 5 controls.

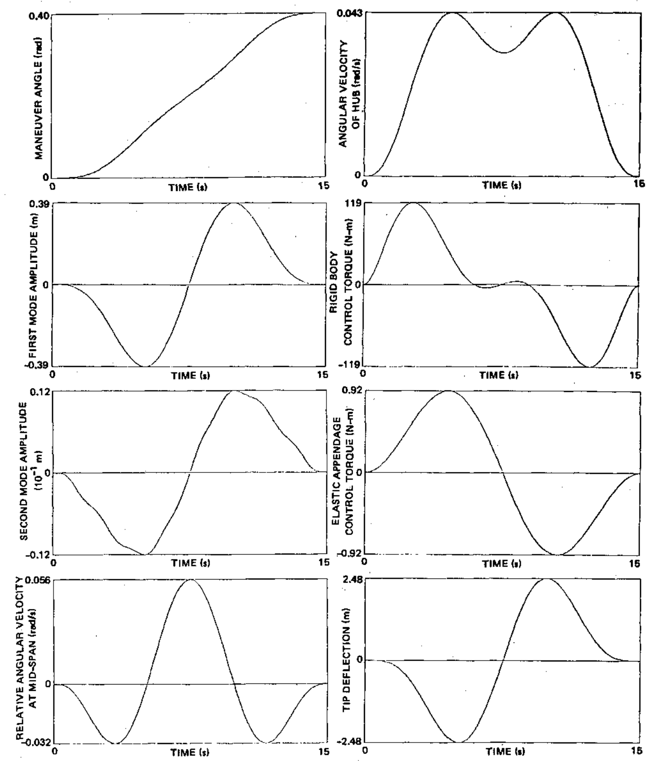

Fig. 5 Case 3, rest-to-rest maneuver, 2 modes, 5 controls, control-rate penalty with $k = 2$.

vector y_5 exists when control rates are penalized. The elements of y_5 correspond to the additionally augmented states.

Referring to Tables 1 and 2, the graphical summaries of the states and controls are discussed qualitatively in what follows.

Case 1 (Fig. 3) presents a terminal controller example maneuver in which two flexible modes are controlled in addition to the rigid body rotation. There are six states in this problem, all of which are specified at the final time. The P, S, and G matrices are, therefore, 6×6 in dimension. The jump discontinuities in the control torques at the initial and final times are characteristic of controllers with no penalities on the control rates in the performance index. The presence of these discontinuities tends to excite higher modes which are not modelled.

Case 2 (Fig. 4) presents a spin-up maneuver with two flexible modes controlled, in which the final angle is free to be determined by the controller. As a result, the P, S, and G matrices have the following dimensions: P (6×6); S (6×5); and $G(5 \times 5)$. On running similar cases with different final angles, it is found that the final angle selected by the terminal controller is the one which minimizes the performance index.

Case 3 (Fig. 5) presents the same maneuver as in Case 1, except that the performance index now includes penalties on the first and second time derivatives of the control torques.[3] Including these penalties allows the control torques and torque-rates to be specified at the initial and final times, thereby eliminating the terminal jump discontinuities in the control profiles. For this particular control problem, the states are augmented to include the controls and first time-derivatives of the controls, while the second time-derivative of the controls becomes the commanded input to the augmented system. Comparing these results with those of Case 1, one finds a much smoother modal amplitude and control torque time history, although the peak torques and tip deflection are somewhat higher than those in Case 1. The dimension of the P, S, and G matrices are all 10×10 for Case 3.

Case 4 (Fig. 6) presents the same maneuver as in Case 2, with the additional penalty on the control rates. As in Case 2, all the states, controls, and control rates are specified at the final time for the terminal controller, except for the maneuver angle, which is not specified at the final time. As a result, the P, S, and G matrices have the following dimensions: P (10×10); S (10×9); and G (9×9). Compared with Case 2, the modal amplitudes and control torque time histories are much smoother, although the peak values are higher. The terminal jump discontinuities in the control torques shown in Case 2 are eliminated in Case 4, making it less susceptible to control spillover.

Table 2 Weighting matrices for example maneuvers

Case no.	Weighting matrices
1, 2	$Q = \mathrm{Diag}[10^{-3}, 10^{-2}, 10^{-3}, 10^{-5}]$ $R = \mathrm{Diag}[10^{-3}, 1]$
3, 5, 6	$Q = \mathrm{Diag}[10^{-3}, 10^{-2}, 10^{-3}, 10^{-5}, 10^{-9}, 10^{-9}, 10^{-9} 10^{-9}]$ $R = \mathrm{Diag}[10^{-3}, 10]$
4	$Q = \mathrm{Diag}[10^{-3}, 10^{-2}, 10^{-3}, 10^{-5}, 10^{-9}, 10^{-9}, 10^{-9} 10^{-9}]$ $R = \mathrm{Diag}[10^{-3}, 1]$

Case 5 (Fig. 7) presents the results of a maneuver in which the weight matrices and final conditions are identical with those of Case 3. The initial conditions are different from those in Case 3, but the time-varying feedback gains are identical. As the plots in Fig. 7 show, the terminal controller has no problem in bringing the system to the required final conditions.

Case 6 (Fig. 8) presents the results of a maneuver in which the actual hub moment of inertia and the appendage are increased by 10% and the appendage mass per unit length decreased by 10%, while the time-varying feedback gains are computed using the nominal structural parameters. Although the structural parameters are changed by only 10%, the overall effect on the entire structure is quite large. Specifically, the first and second mode eigenvalues, ω^2, are increased by 16 and 22%, respectively, while the ratio of the hub moment of inertia to appendage moment of inertia is increased by 22%. However, the plots show that with the

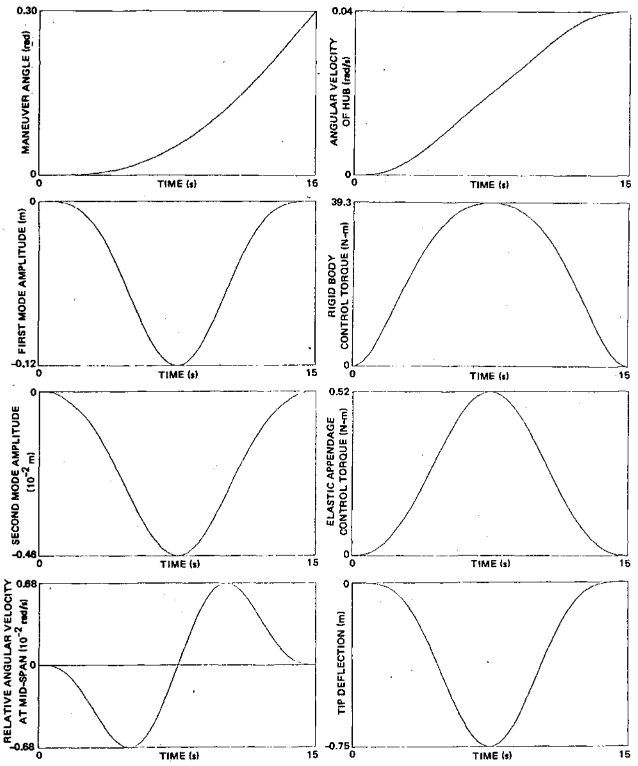

Fig. 6 Case 4, free final angle spin-up maneuver, 2 modes, 5 controls, control-rate penalty with $k = 2$.

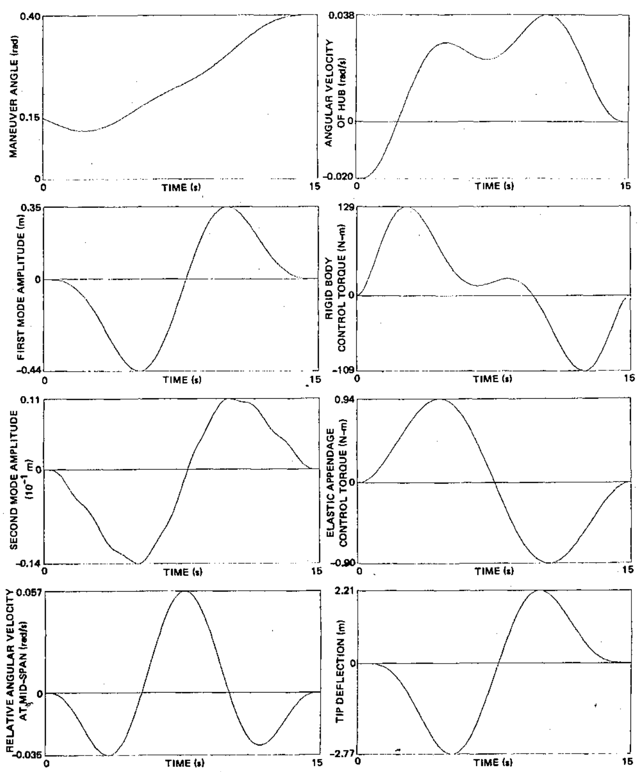

Fig. 7 Case 5, rest-to-rest maneuver with off-nominal initial conditions, 2 modes, 5 controls, control-rate penalty with $k = 2$.

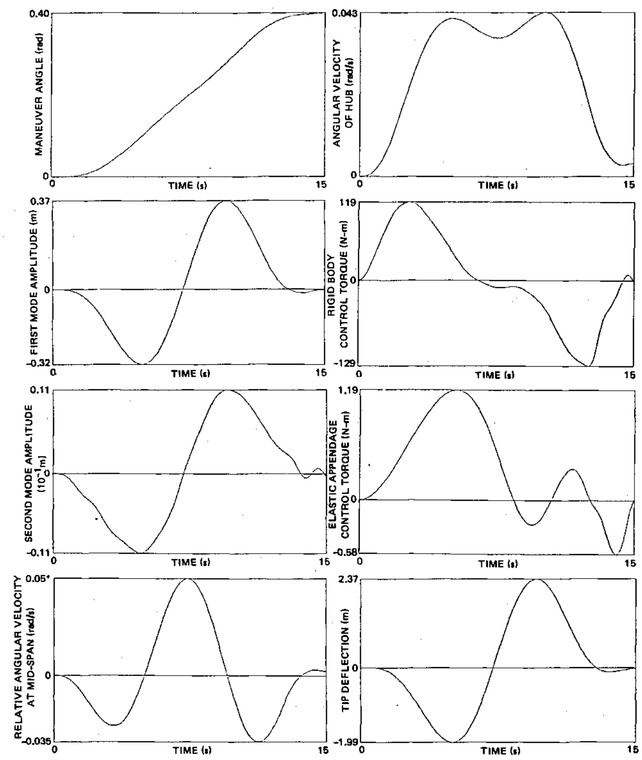

Fig. 8 Case 6, rest-to-rest maneuver with off-nominal structural parameters, 2 modes, 5 controls, control-rate penalty with $k = 2$.

terminal controller, most of the state variables reach their prescribed terminal boundary conditions.

Concluding Remarks

General formulations are given for the closed-loop feedback control of systems with terminal constraints on state and time. Techniques of changing variables were developed for Riccati equations associated with the feedback law. This leads to analytical solutions for the time-varying control gains in which numerical backward integrations for Riccati equations are eliminated. These solutions are particularly attractive for on-board control implementation because computational errors are not accumulated and the solution is computationally efficient.

Results of example maneuvers have been shown which demonstrate the efficiency and validity of the formulation described herein. The use of control-rate penalties in the terminal controller problem has been shown to improve the overall system response. In Cases 1 through 5, results have been compared with the corresponding open-loop maneuvers and have been found to be identical within plotting accuracy, as expected. The closed form solutions for $P(t)$, $S(t)$, and $G(t)$ have also been compared with the backward integrated solutions for verification.

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